Enrollment No: ______________________ **Exam Seat No:**______________________

C.U.SHAH UNIVERSITY

Summer Examination-2020

Subject Name: Functional Analysis

Subject Code: 5SC03FUA1 **Branch: M.Sc. (Mathematics)**

Semester :3 **Date : 29/02/2020** **Time : 02:30 To 05:30** **Marks 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a. Define: Banach space. (01)

b. Prove that every bounded below linear map is one one. (02)

c. In usual notation, prove that $l^p \subset l^\infty$ where $1 \le p < \infty$. (02)

d. State Young's inequality. (02)

Q-2 **Attempt all questions** **(14)**

a. Let $a_j, b_j \in K$ $(j = 1,2, \dots n)$. Let $1 < p, q < \infty$be such that $\frac{1}{p} + \frac{1}{q} = 1$. (07)

Then prove that $\sum_{j=1}^{n} |a_j b_j| \le \left(\sum_{j=1}^{n} |a_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |b_j|^q\right)^{\frac{1}{q}}$

b. Let $a_j, b_j \in K$ $(j = 1,2, \dots n)$ and $1 \le p < \infty$ then prove that $\left(\sum_{j=1}^{n} |a_j +\right.$ *bjp1p ≤ j=1najp1p+ j=1nbjp1p* (07)

OR

Q-2 **Attempt all questions** **(14)**

a. Define norm linear space. (06)

Let $l^p = \left\{ x = (x(j))_{j=1}^{\infty} \mid \|x\|_p = \left(\sum_{j=1}^{\infty} |x(j)|^p\right)^{\frac{1}{p}} < \infty \right\}$ then prove hat $(l^p, \|\cdot\|_p)$ is a norm linear space.

b. State and prove F. Riesz theorem. (05)

c. Let X be a normed linear space. If E_1 is open in X and $E_2 \subset X$ then prove that $E_1 + E_2$ is open in X. (03)

Q-3 **Attempt all questions** **(14)**

a. Let F be a linear map from a norm linear space X to a norm linear space Y then prove the following. (07)
(i) Prove that F is homeomorphism if and only if there exist $\alpha, \beta > 0$ such that $\beta\|x\| \leq \|F(x)\| \leq \alpha\|x\|, \quad \forall\ x \in X$.
(ii) If F is on to and linear homeomorphism then prove that X is complete if and only if Y is complete.

b. Let X be an norm linear space then prove that the following are equivalent (07)
(i)X is a Banach space.
(ii) Every absolutely summable series in $X$is summable.

OR

Q-3 **Attempt all questions** **(14)**

a. Let X be an norm linear space over K and E_1,E_2be nonempty, disjoint subsets of X where $E_1$is open. Then there exists a real hyper plane which separates $E_1$and E_2 in the following sense. For some $f \in X'$ and $t \in R$, $Re\ f(x) < t \leq Re(y), \forall x \in E_1, y \in E_2$. (08)

b. Let X be a complex norm linear space. (06)
(i) If $u: X \to R$ be a linear functional Define $f: X \to \mathbb{C}$ by $f(x) = u(x) - i\,u(ix)$. Then prove that f is a complex linear functional.
(ii) If f is linear functional and define $u(x) = Re\ (f(x))$ then prove that u is a real linear functional and $f(x) = u(x) - i\,u(ix)$.

SECTION – II

Q-4 **Attempt the Following questions** **(07)**

a. Define: Graph of function. (01)

b. Define: Transpose of a function. (01)

c. State Schur's lemma. (01)

d. Let X, Y be normed linear spaces and $f: X \to Y$ be continuous linear map. Then prove that gof is continuous for every $g \in Y'$. (02)

e. State Open mapping theorem. (02)

Q-5 **(14)**

a. Let $1 \leq p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Define $F: l^q \to (l^p)'$ by $F(y) = f_y$ where $y \in l^q$, $f_y \in (l^p)'$ is given by $f_y(x) = \sum_{j=1}^{\infty} x(j)y(j)$. Then prove that F is an on to linear isometry. (07)

b. Let X be a norm linear space and $A \in B(X)$be of finite rank. Then prove that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$. (07)

OR

Q-5 **Attempt all questions** **(14)**

a. Let X be a Banach space, Y be a normed linear space and $\mathcal{A}$ be subset of $B(X, Y)$ such that $\{Ax : A \in \mathcal{A}\}$ is bounded for each $x \in X$. Then prove that $\mathcal{A}$ is a bounded subset of $B(X, Y)$. (07)

b. Let X, Y be normed linear space and $X \neq \{0\}$ then prove that following are equivalent (07)
(i)$BL(X, Y)$ is Banach space.
(ii) Y is Banach space.

Q-6 **Attempt all questions** **(14)**

a. Let X be a normed linear space and $P: X \rightarrow X$ be a projection. Then prove that P is a closed map if and only if $R(P)$ and $Z(P)$ are closed in X. (05)

b. State and prove closed graph theorem (09)

OR

Q-6 **Attempt all Questions** **(14)**

a. Let X be a normed linear space. For every subspace Y of X and every $g \in Y'$. Prove that there is unique Hahn – Banach extension of g to X if and only if X' is strictly convex. (08)

b. (i) Let X be normed linear space. If X_0 is a dense subspace of X then prove that $X_0{}'$ is linear isometry to X'. (06)

(ii) Prove that weak limit of a weakly convergent sequence is unique.